THE BREZIS–NIRENBERG PROBLEM ON $\mathbb{S}^N$, IN SPACES OF FRACTIONAL DIMENSION

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Abstract. We consider the nonlinear eigenvalue problem,

$$-\Delta_{\mathbb{S}^n} u = \lambda u + |u|^{4/(n-2)} u,$$

with $u \in H_0^1(\Omega)$, where Ω is a geodesic ball in $\mathbb{S}^n$ contained in a hemisphere. In dimension 3, Bandle and Benguria proved that this problem has a unique positive solution if and only if

$$\frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} < \lambda < \frac{\pi^2 - \theta_1^2}{\theta_1^2}$$

where θ_1 is the geodesic radius of the ball. For positive radial solutions of this problem one is led to an ODE that still makes sense when n is a real number rather than a natural number. Here we consider precisely that problem with $3 < n < 4$. Our main result is that in this case one has a positive solution if and only if λ is such that

$$\frac{1}{4}[(2\ell_2+1)^2 - (n-1)^2] < \lambda < \frac{1}{4}[(2\ell_1+1)^2 - (n-1)^2]$$

where ℓ_1 (respectively ℓ_2) is the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_\ell^{(2-n)/2}(\cos\theta_1)$ (respectively $\mathrm{P}_\ell^{(n-2)/2}(\cos\theta_1)$) vanishes.

1. Introduction

In 1983, Brezis and Nirenberg [5] considered the nonlinear eigenvalue problem,

$$-\Delta u = \lambda u + |u|^{4/(n-2)} u,$$

with $u \in H_0^1(\Omega)$, where Ω is a bounded smooth domain in $\mathbb{R}^n$, with $n \geq 3$. Among other results, they proved that if $n \geq 4$, there is a positive solution of this problem for all $\lambda \in (0, \lambda_1)$ where λ_1 is the first Dirichlet eigenvalue of Ω. They also proved that if $n = 3$, there is a $\mu(\Omega) > 0$, such that for any $\lambda \in (\mu, \lambda_1)$, the nonlinear eigenvalue problem has a positive solution. Moreover, if Ω is a ball, $\mu = \lambda_1/4$.

For positive radial solutions of this problem in a (unit) ball, on is led to an ODE that still makes sense when n is a real number rather than a natural number. Precisely this problem with $3 \leq n \leq 4$, was considered by E. Janelli [7]. Among other things, Janelli proved that this problem has a positive solution if and only if λ is such that

$$j_{-(n-2)/2,1} < \sqrt{\lambda} < j_{(n-2)/2,1},$$

where $j_{\nu,k}$ denotes the k-th positive zero of the Bessel function J_ν.

Here we consider the nonlinear eigenvalue problem

$$-\Delta_{\mathbb{S}^n} u = \lambda u + |u|^{4/(n-2)} u, \tag{1}$$

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acting on $H_0^1(D)$, where D is a geodesic ball in $\mathbb{S}^n$ contained in a hemisphere. Here $-\Delta_{\mathbb{S}^n}$ denotes the Laplace–Beltrami operator in $\mathbb{S}^n$ and $(n+2)/(n-2)$ is the critical Sobolev exponent. In dimension 3, Bandle and Benguria [2] proved that this problem has a unique positive solution if and only if

$$\frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} < \lambda < \frac{\pi^2 - \theta_1^2}{\theta_1^2},$$

where θ_1 is the geodesic radius of the ball.

As in the Euclidean case, for positive radial solutions of this problem one is led to an ODE that still makes sense when n is a real number. This is the problem we consider in this manuscript, with $2 < n < 4$.

Henceforth, we will only consider positive radial solutions of (1) defined on geodesic caps centered at the north–pole, satisfying Dirichlet boundary conditions, i.e., $u(\theta_1) = 0$. We will denote by θ the azimuthal coordinate of a point on the sphere, with $0 \le \theta \le \theta_1$, and θ_1 being the geodesic radius of the cap. For positive radial functions, (1) reads,

$$-u''(\theta) + (n-1)\cot\theta\, u' = \lambda u + |u|^{4/(n-2)}u, \tag{2}$$

where u is such that $u(\theta_1) = 0$. Here $' \equiv d/d\theta$, etc. As said, the ODE (2) still makes sense when n is not a positive integer. In what follows we will consider n as just being a parameter in equation (2), taking values in $(2, 4)$.

Our main result is the following:

Theorem 1.1. *For any $2 < n < 4$, the boundary value problem (2), in the interval $(0, \theta_1)$, with $u'(0) = u(\theta_1) = 0$ has a positive solution if and only if λ is such that*

$$\frac{1}{4}[(2\ell_2+1)^2 - (n-1)^2] < \lambda < \frac{1}{4}[(2\ell_1+1)^2 - (n-1)^2]$$

where ℓ_1 (respectively ℓ_2) is the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_\ell^{(2-n)/2}(\cos\theta_1)$ (respectively $\mathrm{P}_\ell^{(n-2)/2}(\cos\theta_1)$) vanishes.

In section 2 we begin by showing that $\ell_2 < \ell_1$. That is, the range of existence we obtain above is non-empty. We then show that the upper bound corresponds to the first Dirichlet eigenvalue of the geodesic ball. That is, we show that if λ_1 is the first positive eigenvalue of the boundary value problem

$$-u''(\theta) + (n-1)\cot\theta\, u' = \lambda u$$

with $u(\theta_1) = 0$, then $\lambda_1 = \frac{1}{4}[(2\ell_1+1)^2 - (n-1)^2]$.

In section 3 we show that there are solutions if $\frac{1}{4}[(2\ell_2+1)^2 - (n-1)^2] < \lambda < \frac{1}{4}[(2\ell_1+1)^2 - (n-1)^2]$, and in section 4 we show that there are no solutions if $\lambda \le \frac{1}{4}[(2\ell_2+1)^2 - (n-1)^2]$.

2. Preliminaries

We begin by studying the order of the first positive zeroes of $P_\ell^{\nu}(s)$ and $P_\ell^{-\nu}(s)$ respectively, where $\nu \in (0, 1)$.

Lemma 2.1. *Let $\alpha = (2-n)/2$, with $2 < n < 4$. Let $\theta_1 \in \left(0, \frac{\pi}{2}\right)$ be fixed and choose ℓ_1 (respectively ℓ_2) to be the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_\ell^{(2-n)/2}(\cos\theta_1)$ (respectively $\mathrm{P}_\ell^{(n-2)/2}(\cos\theta_1)$) vanishes. Then $\ell_2 < \ell_1$.*

Proof. Let $y_1 = P_{\ell_1}^{\alpha}(\cos\theta)$ and $y_2 = P_{\ell_2}^{-\alpha}(\cos\theta)$. Then y_1 and y_2 satisfy the equations

$$y_1'' + \cot\theta y_1' + \left(\ell_1(\ell_1+1) - \frac{\alpha^2}{\sin^2\theta}\right) y_1 = 0, \tag{3}$$

and

$$y_2'' + \cot\theta y_2' + \left(\ell_2(\ell_2+1) - \frac{\alpha^2}{\sin^2\theta}\right) y_2 = 0 \tag{4}$$

respectively.

Let $W = y_1'\, y_2 - y_2'\, y_1$ the Wronskian of y_2 and y_1. Then $W' = y_1''\, y_2 - y_2''\, y_1$. Multiplying equation (3) by y_2 and equation (4) by y_1 and substracting it follows that

$$(\sin\theta\, W)' + (\Delta_1 - \Delta_2)\sin\theta\, y_1\, y_2 = 0, \tag{5}$$

where $\Delta_1 = \ell_1(\ell_1+1)$ and $\Delta_2 = \ell_2(\ell_2+1)$. To prove the lemma It suffices to show that $\Delta_1 > \Delta_2$.

Integrating (5) in θ between 0 and θ_1, we get,

$$\sin\theta_1 W(\theta_1) - \lim_{\theta\to 0}\sin\theta\, W(\theta) + (\Delta_1 - \Delta_2)C = 0 \tag{6}$$

where $C = \displaystyle\int_0^{\theta_1} \sin\theta\, y_1(\theta)\, y_2(\theta)\, d\theta > 0$ by hypothesis. Since $W(\theta_1) = 0$, it suffices to show that $\lim_{\theta\to 0}\sin\theta W(\theta) > 0$. The series expansion of the associated Legendre functions around $\theta = 0$ is given by

$$P_\ell^\nu(\cos\theta) = \frac{1}{\Gamma(1-\nu)}\left(\cot\frac{\theta}{2}\right)^{\nu} {}_2F_1\left(-\ell, \ell+1, 1-\nu, \sin^2\frac{\theta}{2}\right), \tag{7}$$

in terms of the hypergeometric function,

$${}_2F_1(\delta,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\delta)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(n+\delta)\Gamma(n+\beta)}{\Gamma(n+\gamma)n!}z^n. \tag{8}$$

From (7) and (8), and using that $-1 < \alpha < 0$, the behavior of $y_1\, y_2$, y_1' and y_2' in a neighborhood of the origin to laeding order is given by

$$y_1 \approx \frac{1}{\Gamma(1-\alpha)}\left(\cot\frac{\theta}{2}\right)^{\alpha},$$

$$y_2 \approx \frac{1}{\Gamma(1+\alpha)}\left(\cot\frac{\theta}{2}\right)^{-\alpha},$$

$$y_1' \approx \frac{\alpha}{\Gamma(1-\alpha)}\left(\cot\frac{\theta}{2}\right)^{\alpha-1}\left(\frac{-1}{2\sin^2\frac{\theta}{2}}\right),$$

and

$$y_2' \approx -\frac{\alpha}{\Gamma(1+\alpha)}\left(\cot\frac{\theta}{2}\right)^{-\alpha-1}\left(\frac{-1}{2\sin^2\frac{\theta}{2}}\right).$$

Using this behavior of $y_1(\theta)$, $y_2(\theta)$, $y_1'(\theta)$, and $y_2'(\theta)$, for small θ, after some calculations we get

$$\lim_{\theta\to 0} \sin\theta\, W(\theta) = \frac{2}{\pi} \sin\left(\frac{\pi(n-2))}{2}\right) > 0, \tag{9}$$

for all $2 < n < 4$. To obtain (9) we have used that $\alpha = (2-n)/2$ and the fact that

$$\Gamma(1+\alpha)\,\Gamma(1-\alpha) = \frac{\pi\alpha}{\sin(\pi\,\alpha)}.$$

□

Lemma 2.2. *Let λ_1 be the first positive eigenvalue of*

$$-u''(\theta) + (n-1)\cot\theta\, u' = \lambda u \tag{10}$$

in the interval $(0, \theta_1)$ with $u'(0) = 0$ and $u(\theta_1) = 0$. Then,

$$\lambda_1 = \frac{1}{4}[(2\ell_1+1)^2 - (n-1)^2],$$

where ℓ_1 is the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_\ell^{(2-n)/2}(\cos\theta_1)$ vanishes.

Proof. Let $\alpha = (2-n)/2$, and set

$$u(\theta) = (\sin\theta)^\alpha\, v(\theta). \tag{11}$$

Then $v(\theta)$ satisfies the equation,

$$v''(\theta) + \frac{\cos\theta}{\sin\theta} v'(\theta) + \left(\lambda_1 + \alpha(\alpha-1) - \frac{\alpha^2}{\sin^2\theta}\right) v = 0. \tag{12}$$

In the particular case when $n = 3$, $\alpha = -1/2$ and this equation becomes,

$$v''(\theta) + \frac{\cos\theta}{\sin\theta} v'(\theta) + \left(\lambda_1 + \frac{3}{4} - \frac{1}{4\,\sin^2\theta}\right) v = 0. \tag{13}$$

whose positive regular solution is given by,

$$v(\theta) = C\,\frac{\sin\left(\sqrt{1+\lambda_1}\,\theta\right)}{\sqrt{\sin\theta}}. \tag{14}$$

Hence, in this case,

$$u(\theta) = C\,\frac{\sin\left(\sqrt{1+\lambda_1}\,\theta\right)}{\sin\theta}. \tag{15}$$

Imposing the boundary condition $u(\theta_1) = 0$, in the case $n = 3$, we find that,

$$\lambda_1(\theta_1) = \frac{\pi^2 - \theta_1^2}{{\theta_1}^2}. \tag{16}$$

Now, for any $3 < n < 4$ the solutions of (13) are $P_\ell^\alpha(\cos\theta)$ and $P_\ell^{-\alpha}(\cos\theta)$, with

$$\alpha = (2-n)/2, \tag{17}$$

and ℓ the positive root of

$$\ell(\ell+1) = \lambda_1 + \alpha(\alpha-1), \tag{18}$$

that is,

$$\ell = \frac{1}{2}\left(\sqrt{4\lambda_1 + (n-1)^2} - 1\right).$$

Taking into account (7) and (8) we see that the regular solution of (10) is given by

$$u(\theta) = \sin^{\alpha}\theta\, P_{\ell}^{\alpha}(\cos\theta). \tag{19}$$

Finally, the boundary conditions $u(\theta_1) = 0$ and $u(\theta) > 0$ if $0 \leq \theta < \theta_1$ imply that $\ell = \ell_1$, and so

$$\lambda_1 = \frac{1}{4}[(2\ell_1 + 1)^2 - (n-1)^2].$$

Here, ℓ_1 is the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_{\ell}^{(2-n)/2}(\cos\theta_1)$ vanishes. □

3. Existence of solutions

Let D be a geodesic ball on $\mathbb{S}^n$. If n is a natural number, the solutions of

$$\begin{cases} -\Delta_{\mathbb{S}^n} u = \lambda u + u^p & \text{on} \quad D \\ \mathrm{u} > 0 & \text{on} \quad D \\ \mathrm{u} = 0 & \text{on} \quad \partial D, \end{cases} \tag{20}$$

where $p = \frac{n+2}{n-2}$ correspond to minimizers of

$$Q_\lambda(u) = \frac{\displaystyle\int_D (\nabla u)^2 q^{n-2}\, dx - \lambda \int_D u^2 q^n\, dx}{\left(\displaystyle\int_D u^{\frac{2n}{n-2}} q^n\, dx\right)^{\frac{n-2}{n}}}. \tag{21}$$

Here $q(x) = \frac{2}{1+|x|^2}$, so that $ds = q(x)dx$ is the line element of $\mathbb{S}^n$; and $x \in D'$, where D' is the projection of the stereographic ball.

If u is radial, then even for fractional n we can write

$$Q_\lambda(u) = \frac{\omega_n \displaystyle\int_0^R r^{n-1} q(r)^{n-2} u_\epsilon'(r)^2\, dr - \lambda\omega_n \int_0^R r^{n-1} q(r)^n u^2(r)\, dr}{\left(\omega_n \displaystyle\int_0^R r^{n-1} q^n u(r)^{\frac{2n}{n-2}}\, dr\right)^{\frac{n-2}{n}}}. \tag{22}$$

Here R corresponds to the stereographic projection of θ_1.

As in [2], let

$$S_{p,\lambda}(D) = \inf_{\substack{u\in H_0^1 \\ ||u||_{p+1}=1}} \{||\nabla u||_2^2 - \lambda ||u||_2^2\}, \tag{23}$$

so that $S_\lambda \leq Q_\lambda(u)$, and let

$$S = \inf_{\substack{u\in H_0^1 \\ ||u||_{p+1}=1}} ||\nabla u||_2^2. \tag{24}$$

By the Brezis–Lieb compactness lemma [4], it is known that in $\mathbb{R}^n$, if there is a function that satisfies $Q_\lambda(u) < S$, then the minimizer for Q_λ is attained. The minimizer is positive and satisfies the Brezis–Nirenberg equation. Bandle and Peletier [3] proved that for domains in $\mathbb{S}^n$ contained in the hemisphere, the Brezis–Lieb lemma still holds.

Lemma 3.1. *Let 2¡n¡4 and*

$$\frac{1}{4}[(2\ell_2+1)^2-(n-1)^2] < \lambda < \frac{1}{4}[(2\ell_1+1)^2-(n-1)^2],$$

where ℓ_1 (respectively ℓ_2) is the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_\ell^{(2-n)/2}(\cos\theta_1)$ (respectively $\mathrm{P}_\ell^{(n-2)/2}(\cos\theta_1)$) vanishes. Then there is a positive solution to

$$-u''(\theta)+(n-1)\cot\theta\, u' = \lambda u \tag{25}$$

with $u'(0)=u(\theta_1)=0$.

Proof. It suffices to show that there exists $u \in H_0^1(D)$ such that $Q_\lambda(u) < S$

Let φ be a smooth function such that $\varphi(0)=1$, $\varphi'(0)=0$ and $\varphi(R)=0$, where R is the stereographic projection of θ_1. For $\epsilon > 0$, let

$$u_\epsilon(r) = \frac{\varphi(r)}{(\epsilon+r^2)^{\frac{n-2}{2}}}. \tag{26}$$

We claim that for ϵ small enough, $Q_\lambda(u_\epsilon) \leq S$. In the next three claims we compute $||\nabla u_\epsilon||_2^2$, $||u_\epsilon||_{p+1}^2$ and $||u_\epsilon||_2^2$.

Claim 3.2.

$$\begin{aligned}\omega_n\int_0^R r^{n-1}q(r)^{n-2}u_\epsilon'(r)^2\,dr =& \,\omega_n\int_0^R \varphi'(r)^2 r^{3-n}q^{n-2}\,dr - \omega_n(n-2)^2\int_0^R \varphi(r)^2r^{3-n}q^{n-1}\,dr\\ &+\omega_n n(n-2)2^{n-2}D_n\epsilon^{\frac{2-n}{2}} + \mathcal{O}(\epsilon^{\frac{4-n}{2}}),\end{aligned} \tag{27}$$

where

$$D_n = \frac{1}{2}\frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)}, \qquad \omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{28}$$

Proof. Let

$$I(\epsilon) = \omega_n\int_0^R r^{n-1}q(r)^{n-2}u_\epsilon'(r)^2\,dr.$$

Then

$$I(\epsilon) = \omega_n\int_0^R r^{n-1}q^{n-2}\left(\frac{\varphi'^2}{(\epsilon+r^2)^{n-2}} - \frac{2(n-2)r\varphi\varphi'}{(\epsilon+r^2)^{n-1}} + \frac{r^2\varphi^2(n-2)^2}{(\epsilon+r^2)^n}\right)dr. \tag{29}$$

Integrating by parts the term with $\varphi\varphi'$, we obtain $I(\epsilon) = I_1+I_2+I_3$, where

$$I_1(\epsilon) = \omega_n\int_0^R r^{n-1}q^{n-2}\frac{\varphi'^2}{(\epsilon+r^2)^{n-2}}\,dr;$$

$$I_2(\epsilon) = \omega_n(n-2)^2\int_0^R r^nq^{n-3}q'\,\frac{\varphi^2}{(\epsilon+r^2)^{n-1}}\,dr;$$

and

$$I_3(\epsilon) = \omega_n (n-2) n\epsilon \int_0^R q^{n-2} r^{n-1} \frac{\varphi^2}{(\epsilon + r^2)^n}.$$

We begin by showing that

$$I_1(\epsilon) = \omega_n \int_0^R r^{3-n} q^{n-2} \varphi'^2 \, dr + \mathcal{O}(\epsilon).$$

Notice that

$$I_1(0) = \omega_n \int_0^R r^{3-n} q^{n-2} \varphi'^2 \, dr$$

converges for $n < 4$. It suffices to show that $I_1(\epsilon) - I_1(0) = \mathcal{O}(\epsilon)$. We can write

$$I_1(\epsilon) - I_1(0) = \omega_n \int_0^R r^{n-1} q^{n-2} \varphi'^2 \int_0^\epsilon \frac{n-2}{(a + r^2)^{n-1}} \, da \, dr.$$

But

$$\int_0^R q^{n-2} r^{n-1} \frac{\varphi'^2}{(a + r^2)^{n-1}} \, dr \leq \int_0^R 2^{n-2} r^{3-n} \, dr,$$

which converges if $n < 4$, thus yielding the desired result.

Next let us consider I_2. We will show that

$$I_2(\epsilon) = -\omega_n (n-2)^2 \int_0^R q^{n-1} \varphi^2 r^{3-n} \, dr + \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

Notice that $q' = -q^2 r$, so that

$$I_2(\epsilon) = -\omega_n (n-2)^2 \int_0^R q^{n-1} r^{n+1} \frac{\varphi^2}{(\epsilon + r^2)^{n-1}} \, dr.$$

As in the previous integral, let $I_2(\epsilon) = I_2(0) + I_2(\epsilon) - I_2(0)$. Then it suffices to show that $I_2(\epsilon) - I_2(0) = \mathcal{O}(\epsilon^{\frac{4-n}{2}})$. We can write

$$\begin{aligned} I_2(\epsilon) - I_2(0) =& \omega_n (n-2)^2 \int_0^R \varphi^2 q^{n-1} r^{n+1} \left(\frac{1}{r^{2n-2}} - \frac{1}{(\epsilon + r^2)^{n-1}} \right) dr \\ =& \omega_n (n-2)^2 \int_0^R q^{n-1} r^{n+1} \left[(\varphi^2 - 1) + 1 \right] \int_0^\epsilon \frac{n-1}{(a + r^2)^n} \, da \, dr. \end{aligned} \tag{30}$$

Let

$$I_{21}(\epsilon) = \int_0^R q^{n-1} r^{n+1} \int_0^\epsilon \frac{n-1}{(a + r^2)^n} \, da \, dr,$$

and

$$I_{22}(\epsilon) = \int_0^R q^{n-1} r^{n+1} (\varphi^2 - 1) \int_0^\epsilon \frac{n-1}{(a + r^2)^n} \, da \, dr.$$

Then, since $q^n \leq 2^n$, and making the change of variables $r = s\sqrt{a}$, it follows that

$$I_{21}(\epsilon) \leq 2^{n-1}(n-1)\int_0^\epsilon a^{\frac{2-n}{2}}\int_0^\infty \frac{s^{n+1}}{(1+s^2)^n}\,ds\,da.$$

The inner integral converges if $n > 2$, so it follows that

$$I_{21}(\epsilon) = \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

Also, since by hypothesis $\varphi(0) = 1$ and $\varphi'(0) = 0$, it follows that $\varphi^2 - 1 \leq Cr^2$. Thus,

$$I_{22}(\epsilon) \leq C2^{n-1}(n-1)\int_0^\epsilon \int_0^R r^{3-n}\,dr\,da.$$

The inner integral converges if $n < 4$, so it follows that $I_{22}(\epsilon) = \mathcal{O}(\epsilon)$. In particular, since $n \geq 2$, $I_{22}(\epsilon) = \mathcal{O}(\epsilon^{\frac{4-n}{2}})$ and

$$I_2(\epsilon) - I_2(0) = \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

Finally, we must show that

$$I_3(\epsilon) = \omega_n n(n-2)2^{n-2}D_n\epsilon^{\frac{2-n}{2}} + \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

Writing

$$q^{n-2}\varphi^2 = q^{n-2}(\varphi^2-1) + (q^{n-2} - 2^{n-2}) + 2^{n-2},$$

we have that $I_3 = \omega_n(n-2)n(I_{31} + I_{32} + I_{33})$, where

$$I_{31} = \int_0^R \frac{\epsilon\, r^{n-1}q^{n-2}(\varphi^2-1)}{(\epsilon+r^2)^n}\,dr;$$

$$I_{32} = \int_0^R \frac{\epsilon\, r^{n-1}(q^{n-2}-2^{n-2})}{(\epsilon+r^2)^n}\,dr;$$

and

$$I_{33} = 2^{n-2}\int_0^R \frac{\epsilon\, r^{n-1}}{(\epsilon+r^2)^n}\,dr.$$

As before, since $\varphi^2 - 1 \leq Cr^2$, it follows that

$$I_{31} \leq C2^{n-2}\epsilon\int_0^R \frac{r^{n+1}}{(\epsilon+r^2)^n}\,dr.$$

Letting $r = s\sqrt{\epsilon}$, it follows that

$$\int_0^R \frac{r^{n+1}}{(\epsilon+r^2)^n}\,dr \leq \epsilon^{\frac{2-n}{2}}\int_0^\infty \frac{s^{n+1}}{(1+s^2)^n}\,ds = \mathcal{O}(\epsilon^{\frac{2-n}{2}}), \tag{31}$$

since the integral converges for all $n > 2$. Thus,

$$I_{31} = \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

Similarly, and since if $0 \leq r \leq R$ then $2^{n-2} - q^{n-2} \leq 2^{n-2}A(R)r^2$, with $A(R) = (n-2)(1+R^2)^{n-3}$, we have that

$$|I_{32}| \leq 2^{n-2} A(R)\epsilon \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n}\, dr = \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

Finally, making the change of variables $r = s\sqrt{\epsilon}$, it follows that

$$I_{33} = 2^{n-2}\epsilon^{\frac{2-n}{2}} \left[\int_0^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds - \int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds \right].$$

But

$$\int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds \leq \int_{\frac{R}{\sqrt{\epsilon}}}^\infty s^{-n-1}\, ds = \mathcal{O}(\epsilon^{\frac{n}{2}}). \tag{32}$$

Moreover, notice that making the change of variables $u = s^2$, we can write

$$\int_0^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds = \frac{1}{2}\int_0^\infty \frac{u^{\frac{n}{2}-1}}{(1+u)^n}\, du = \frac{1}{2}\frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)} = D_n. \tag{33}$$

Here we have used the standard integral

$$\int_0^\infty \frac{x^{k-1}}{(1+x)^{k+m}}\, dx = \frac{\Gamma(k)\Gamma(m)}{\Gamma(k+m)} \tag{34}$$

(see, e.g., [6], equation 856.11, page 213), which holds for all $m,\ k\ > 0$. Thus,

$$I_{33} = 2^{n-2}\epsilon^{\frac{2-n}{2}} D_n + \mathcal{O}(\epsilon).$$

This yields the desired estimate for I_3. □

Claim 3.3.

$$\omega_n \int_0^R r^{n-1} q^n u^2\, dr = \omega_n \int_0^R q^n r^{3-n} \varphi^2\, dr + \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

Proof. Let

$$J(\epsilon) = \omega_n \int_0^R r^{n-1} q^n \frac{\varphi^2}{(\epsilon + r^2)^{n-2}}\, dr.$$

Then

$$J(0) = \omega_n \int_0^R q^n r^{3-n} \varphi^2\, dr.$$

Thus, it suffices to show that $|J(\epsilon) - J(0)| = \mathcal{O}(\epsilon^{\frac{4-n}{2}})$. We can write

$$|J(\epsilon) - J(0)| = \omega_n \int_0^R q^n \left[(\varphi^2 - 1) + 1 \right] r^{n-1} \int_0^\epsilon \frac{n-2}{(a + r^2)^{n-1}}\, da\, dr.$$

Let

$$J_1(\epsilon) = \int_0^\epsilon \int_0^R \frac{q^n r^{n-1}}{(a + r^2)^{n-1}}\, dr\, da, \tag{35}$$

and

$$J_2(\epsilon) = \int_0^R (\varphi^2 - 1) q^n r^{n-1} \int_0^\epsilon \frac{1}{(a+r^2)^{n-1}}\, da\, dr.$$

Making the change of variables $r = s\sqrt{a}$ in the inner integral of equation (35) we have that

$$J_1(\epsilon) \leq 2^n \int_0^\epsilon a^{\frac{2-n}{2}} \int_0^\infty \frac{s^{n-1}}{(1+s^2)^{n-1}}\, ds\, da.$$

Since $2 < n < 4$ it follows that $J_1(\epsilon) = \mathcal{O}(\epsilon^{\frac{4-n}{2}})$.

Moreover, since $\varphi^2 - 1 \leq Cr^2$, it follows that if $n < 4$, then

$$J_2(\epsilon) \leq C \int_0^R q^n r^{n+1} \int_0^\epsilon \frac{1}{(a+r^2)^{n-1}}\, da\, dr \leq C 2^n\, \epsilon \int_0^R r^{3-n}\, dr = \mathcal{O}(\epsilon).$$

Thus, and since $2 < n < 4$, it follows that $|J(\epsilon) - J(0)| = \mathcal{O}(\epsilon^{\frac{4-n}{2}})$.

□

Claim 3.4.

$$\left(\omega_n \int_0^R r^{n-1} q^n u_\epsilon^{\frac{2n}{n-2}}\, dr\right)^{\frac{n-2}{n}} = \omega_n^{\frac{n-2}{n}} 2^{n-2} \epsilon^{\frac{2-n}{2}} D_n^{\frac{n-2}{n}} + \mathcal{O}(\epsilon^{\frac{4-n}{2}}),$$

where

$$D_n = \frac{1}{2} \frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)}.$$

Proof. Let

$$K(\epsilon) = \omega_n \int_0^R r^{n-1} q^n u_\epsilon^{\frac{2n}{n-2}}\, dr = \omega_n \int_0^R r^{n-1} q^n \frac{\varphi^{\frac{2n}{n-2}}}{(\epsilon + r^2)^n}\, dr.$$

Then, and since $q^n \varphi^{\frac{2n}{n-2}} = q^n(\varphi^{\frac{2n}{n-2}} - 1) + (q^n - 2^n) + 2^n$, we can write $K(\epsilon) = \omega_n(K_1(\epsilon) + K_2(\epsilon) + K_3(\epsilon))$, where

$$K_1(\epsilon) = \int_0^R \frac{q^n r^{n-1}}{(\epsilon + r^2)^n} (\varphi^{\frac{2n}{n-2}} - 1)\, dr;$$

$$K_2(\epsilon) = \int_0^R \frac{r^{n-1}(q^n - 2^n)}{(\epsilon + r^2)^n}\, dr;$$

and

$$K_3(\epsilon) = 2^n \int_0^\epsilon \frac{r^{n-1}}{(\epsilon + r^2)^n}\, dr.$$

Since $\varphi(0) = 1$ and $\varphi'(0) = 1$ it follows that $\varphi^{\frac{2n}{n-2}} - 1 \leq Cr^2$. Thus, making the change of variables $r = s\sqrt{\epsilon}$, and since $n > 2$, it follows that

$$K_1(\epsilon) \leq C 2^n \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n}\, dr \leq C 2^n \epsilon^{\frac{2-n}{2}} \int_0^\infty \frac{s^{n+1}}{(1+s^2)^n}\, ds = \mathcal{O}(\epsilon^{\frac{2-n}{2}}). \qquad (36)$$

In order to obtain an estimate for $K_2(\epsilon)$, notice that if $0 \le r \le R$, then $0 \le 2^n - q^n \le 2^n A(R) r^2$, where $A(R) = n(1+R^2)^{n-1}$. Thus,

$$|K_2(\epsilon)| \le 2^n A(R) \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n}\, dr.$$

As before, we can make the change of variables $r = s\sqrt{\epsilon}$ to obtain

$$|K_2(\epsilon)| \le 2^n A(R) \epsilon^{\frac{2-n}{2}} \int_0^R \frac{s^{n+1}}{(1+s^2)^n}\, ds = \mathcal{O}(\epsilon^{\frac{2-n}{2}}). \tag{37}$$

Finally, we will show that

$$K_3(\epsilon) = 2^n \epsilon^{\frac{-n}{2}} D_n + \mathcal{O}(1). \tag{38}$$

In fact, making the change of variables $r = s\sqrt{\epsilon}$ we have that

$$K_3(\epsilon) = 2^n \epsilon^{\frac{-n}{2}} \left(\int_0^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds - \int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds \right).$$

But by equations (33) and (32) it follows that

$$\int_0^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds = D_n,$$

and

$$\int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds = \mathcal{O}(\epsilon^{\frac{n}{2}}).$$

It follows from equations (36), (37) and (38) that

$$K(\epsilon) = 2^n \omega_n \epsilon^{\frac{-n}{2}} D_n + \mathcal{O}(\epsilon^{\frac{n}{2}}),$$

and so

$$K(\epsilon)^{\frac{n-2}{n}} = \omega_n^{\frac{n-2}{n}} 2^{n-2} \epsilon^{\frac{2-n}{2}} D_n^{\frac{n-2}{n}} + \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

□

Recall that our goal is to show that if $\lambda > \frac{1}{4}[(2\ell_2+1)^2 - (n-1)^2]$, then

$$Q_\lambda(u_\epsilon) = \frac{\int (\nabla u_\epsilon)^2 q^{n-2}\, dx - \lambda \int u_\epsilon^2 q^n\, dx}{\left(\int u_\epsilon^{\frac{2n}{n-2}} q^n\, dx \right)^{\frac{n-2}{n}}} < S \tag{39}$$

where S is the Sobolev critical constant.

From the estimates obtained in Claim 3.2, Claim 3.3 and Claim 3.4 it follows that

$$Q_\lambda(u_\epsilon) = n(n-2)(\omega_n D_n)^{\frac{2}{n}} + \epsilon^{\frac{n-2}{2}} C_n \left[\int_0^R r^{3-n} \left(q^{n-2}\varphi'^2 - (n-2)^2 q^{n-1}\varphi^2 - \lambda q^n \varphi^2 \right) dr \right] + \mathcal{O}(\epsilon), \tag{40}$$

where $C_n = \omega_n^{\frac{2}{n}} 2^{2-n} D_n^{\frac{2-n}{n}}$.

Notice that

$$n(n-2)(\omega_n D_n)^{\frac{2}{n}} = \pi n(n-2) \left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)} \right)^{\frac{2}{n}},$$

which is precisely the Sobolev critical constant S (see, e.g., [10], with $p = 2$, $m = n$ and $q = \frac{2n}{n-2}$).

Let

$$T(\varphi) = \int_0^R r^{3-n} \left(q^{n-2}\varphi'^2 - (n-2)^2 q^{n-1}\varphi^2 - \lambda q^n \varphi^2 \right) dr.$$

It suffices to show that $T(\varphi)$ is negative if $\lambda > \frac{1}{4}[(2\ell_2+1)^2 - (n-1)^2]$. In order to conclude the proof we choose $\varphi = \varphi_1$, where φ_1 is the minimizer of

$$M(\varphi) = \int_0^R r^{3-n} \left(q^{n-2}\varphi'^2 - (n-2) q^{n-1}\varphi^2 \right) dr$$

subject to the constraint

$$\int_0^R r^{3-n} q^n \varphi^2 \, dr = 1.$$

The minimizer of $M(\varphi)$, φ_1, satisfies the Euler equation

$$-\frac{d}{dr}\left(r^{3-n} q^{n-2} \varphi_1' \right) - (n-2) r^{3-n} q^{n-1} \varphi_1 = \mu q^n r^{3-n} \varphi_1. \tag{41}$$

Multiplying (41) by $\varphi_1(r)$ and integrating between 0 and R we get, after integrating by parts,

$$\int_0^R r^{3-n} q^{n-2} \varphi_1'^2 \, dr - (n-2) \int_0^R r^{3-n} q^{n-1} \varphi_1^2 \, dr = \mu \int_0^R q^n r^{3-n} \varphi_1^2 \, dr.$$

Thus, since $\int_0^R q^n r^{3-n} \varphi_1^2 \, dr = 1$, $M(\varphi_1) = \mu$; hence

$$T(\varphi_1) = M(\varphi_1) - \lambda = \mu - \lambda < 0$$

if $\lambda > \mu$.

It suffices to show that $\mu = \frac{1}{4}\left[(2\ell_2+1)^2 - (n-1)^2 \right]$, where ℓ_2 is the first positive value for which the associated Legendre function $P_\ell^{\frac{(n-2)}{2}}(\cos\theta_1)$ vanishes.

Changing coordinates (setting $r = \tan\theta/2$, so that $q = 2\cos^2\theta/2$) and letting

$$\varphi_1(\theta) = \sin^b \left(\frac{\theta}{2} \right) \sin^a(\theta)\, v(\theta),$$

where $b = 2n-4$ and $a = \frac{1}{2}(6-3n)$ we obtain the equation for v

$$\ddot{v}(\theta) + \cot\theta\, \dot{v}(\theta) + \left(\mu + \frac{n(n-2)}{4} - \frac{(n-2)^2}{4\sin^2\theta} \right) v = 0, \tag{42}$$

with boundary condition $v(\theta_1) = 0$.

Remark 3.5. *This equation is the same equation that determines the first Dirichlet eigenvalue of the original problem (equation 12). We choose a and b precisely so that these two equations coincide.*

The solutions of equation (42) are P_ℓ^α and $P_\ell^{-\alpha}$, where $\alpha = \frac{2-n}{2}$ and $\ell(\ell+1) = \mu + \frac{n(n-2)}{4}$. That is, $\ell = \frac{1}{2}\left(\sqrt{1+4\mu-4\alpha+4\alpha^2}-1\right)$, and so

$$\mu = \frac{1}{4}\left[(2\ell+1)^2 - (n-1)^2\right].$$

It follows that φ_1 is of the form

$$\varphi_1 = \sin^b\left(\frac{\theta}{2}\right)\sin^a\theta(AP_\ell^\alpha + BP_\ell^{-\alpha}),$$

where the choice of A and B must ensure the regularity of the solution. Notice that from the definition of a and b we have that $a+b = (n-2)/2$. Moreover, $\alpha = (2-n)/2$. Since $2 < n$, we see that in order to have regular solutions at the origin we have to choose $A = 0$. Finally, to satisfy the boundary condition $u(\theta_1) = 0$ we must choose $\ell = \ell_2$. which finishes the proof of the lemma. ☐

4. Nonexistence of solutions

In this section we use a Rellich–Pohozaev [8, 9] type argument to prove the nonexistence of regular positive solutions of the Boundary Value Problem

$$-u'' - (n-1)\cot\theta\, u' = u^p + \lambda\, u \tag{43}$$

in the interval $(0, \theta_1)$, with boundary conditions $u'(0) = 0$, $u(\theta_1) = 0$ for a sharp range of values of λ. Here $2 < n < 4$ and $p = (n+2)/(n-2)$ is the critical Sobolev exponent. Our main result in this section is the following Lemma.

Lemma 4.1. *Let ℓ_2 be the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_\ell^{(n-2)/2}(\cos\theta_1)$ vanishes. Then if*

$$\lambda \le \frac{1}{4}[(2\ell_2+1)^2 - (n-1)^2],$$

there are no positive solutions of

$$-\frac{(\sin^{n-1}\theta\, u')'}{\sin^{n-1}\theta} = u^p + \lambda u, \tag{44}$$

with boundary conditions $u'(0) = 0$, and $u(\theta_1) = 0$.

Remark 4.2. *Notice that we have recast equation (43) in the form (44) which is more suitable in our proof.*

Proof. Multiplying equation (44) by $g(\theta)u'(\theta)\sin^{2n-2}\theta$, where $g(\theta)$ is a sufficiently smooth, nonnegative function defined in the interval $(0, \theta_1)$ satisfying the boundary conditions $g(0) = g'(0) = 0$, we obtain

$$-\int_0^{\theta_1}(\sin^{n-1}\theta\, u')'\, u'\, g\, \sin^{n-1}\theta\, d\theta = \int_0^{\theta_1}\left(\frac{u^{p+1}}{p+1}\right)' g\, \sin^{2n-2}\theta\, d\theta + \lambda\int_0^{\theta_1}\left(\frac{u^2}{2}\right)' g\, \sin^{2n-2}\theta\, d\theta.$$

Integrating all the terms by parts, using the boundary conditions, we have that

$$\begin{aligned}\int_0^{\theta_1} u'^2 \left(\frac{g'}{2}\sin^{2n-2}\theta\right) d\theta + \int_0^{\theta_1} \frac{u^{p+1}}{p+1}\left(g'\sin^{2n-2}\theta + g(2n-2)\sin^{2n-3}\theta\cos\theta\right) d\theta \\ + \lambda\int_0^{\theta_1}\frac{u^2}{2}\left(g'\sin^{2n-2}\theta + g(2n-2)\sin^{2n-3}\theta\cos\theta\right) d\theta = \frac{1}{2}\sin^{2n-2}\theta_1 u'(\theta_1)^2 g(\theta_1).\end{aligned} \tag{45}$$

On the other hand, setting $h = \frac{1}{2}g'\,\sin^{n-1}\theta$ and multiplying equation (44) by $h(\theta)\,u(\theta)\sin^{n-1}(\theta)$ we obtain

$$-\int_0^{\theta_1}(\sin^{n-1}\theta u')' hu\, d\theta = \int_0^{t_1} hu^{p+1}\sin^{n-1}\theta\, d\theta + \lambda\int_0^{\theta_1} hu^2\sin^{n-1}\theta\, d\theta.$$

Integrating by parts we obtain

$$\begin{aligned}\int_0^{\theta_1} u'^2 h\sin^{n-1}\theta\, d\theta = \int_0^{\theta_1} u^{p+1}h\sin^{n-1}\theta\, d\theta \\ + \int_0^{\theta_1} u^2\left(\lambda h\sin^{n-1}\theta + \frac{1}{2}h''\sin^{n-1}\theta + \frac{1}{2}h'(n-1)\sin^{n-2}\theta\cos\theta\right) d\theta.\end{aligned} \tag{46}$$

Notice that by our choice of h, the coefficient of u'^2 in equation (45) is the same as the coefficient of u'^2 in equation (46). Finally, subtracting equation (45) from equation (46) we obtain

$$\frac{1}{2}\sin^{2n-2}\theta_1 u'(\theta_1)^2 g(\theta_1) = \int_0^{\theta_1} B\,u^{p+1}\, d\theta + \int_0^{\theta_1} A\,u^2\, d\theta, \tag{47}$$

where

$$\begin{aligned}A \equiv \lambda\left(h\sin^{n-1}\theta + \frac{1}{2}g'\sin^{2n-2}\theta + g(n-1)\sin^{2n-3}\theta\cos\theta\right) \\ + \frac{1}{2}h''\sin^{n-1}\theta + \frac{1}{2}h'(n-1)\sin^{n-2}\theta\cos\theta,\end{aligned} \tag{48}$$

and

$$B \equiv h\sin^{n-1}\theta + \frac{g'\sin^{2n-2}\theta}{p+1} + \frac{(2n-2)g\sin^{2n-3}\theta\cos\theta}{p+1}. \tag{49}$$

Since by hypothesis $g(\theta_1) \geq 0$, it follows that the left hand side of equation (47) is nonnegative. In the sequel (see the Claim 4.3 and the Lemma 4.4 below), we show that for any

$$\lambda \leq \frac{1}{4}[(2\ell_2+1)^2 - (n-1)^2],$$

there exists a choice of g so that $A \equiv 0$, and B is negative. That is, we will show that for that range of λ's the right hand side of equation (47) is negative, thus obtaining a contradiction. □

Substituting $h = \frac{1}{2}g'\,\sin^{n-1}\theta$ in equation (48) we obtain

$$\begin{aligned}A = \sin^{2n-2}\theta\Bigg[\frac{g'''}{4} + \frac{3}{4}g''(n-1)\cot\theta \\ + g'\left(\frac{(n-1)(2n-3)\cot^2\theta}{4} - \frac{n-1}{4} + \lambda\right) + \lambda g(n-1)\cot\theta\Bigg].\end{aligned} \tag{50}$$

Finally, making the change of variables $g = f/\sin^2\theta$ we obtain

$$\begin{aligned} A = \sin^{2n-4}\theta \Bigg[\frac{f'''}{4} + \frac{3}{4}(n-3)\cot\theta f'' + f'\left(\frac{(n-3)(2n-11)}{4}\cot^2\theta + \frac{7-n}{4} + \lambda\right) \\ +f\Big((n-3)(4-n)\cot^3\theta + 2(n-3)\cot\theta + \lambda(n-3)\cot\theta\Big)\Bigg]. \end{aligned} \tag{51}$$

Claim 4.3. *For any $2 < n < 4$, the function*

$$z(\theta) = \sin^{4-n}\theta P_\ell^\alpha(\cos\theta)P_\ell^{-\alpha}(\cos\theta),$$

with $\alpha = (2-n)/2$ and $\ell = \frac{1}{2}\left(\sqrt{4\lambda + (n-1)^2} - 1\right)$, is a solution of

$$\begin{aligned} \frac{f'''}{4} + \frac{3}{4}(n-3)\cot\theta f'' + f'\left(\frac{(n-3)(2n-11)}{4}\cot^2\theta + \frac{7-n}{4} + \lambda\right) \\ + f\Big((n-3)(4-n)\cot^3\theta + 2(n-3)\cot\theta + \lambda(n-3)\cot\theta\Big) = 0. \end{aligned} \tag{52}$$

Proof. Let $y_1(\theta) = P_\ell^\alpha(\cos\theta)$ and $y_2(\theta) = P_\ell^{-\alpha}(\cos\theta)$. Then y_1 and y_2 are solutions to

$$y''(\theta) + \cot\theta y'(\theta) + k(\theta)y(\theta) = 0, \tag{53}$$

where

$$k(\theta) = \ell(\ell+1) - \frac{\alpha^2}{\sin^2\theta}. \tag{54}$$

Let $v(\theta) = y_1(\theta)\, y_2(\theta)$. Then, it follows from (53) that

$$y_1''y_2 + y_2''y_1 = -\cot\theta v' - 2kv,$$

which in turn implies

$$v'' = -2kv - \cot\theta v' + 2y_1'y_2'.$$

Similarly, and since

$$y_1''y_2' + y_1'y_2'' = -2\cot\theta y_1'y_2' - kv',$$

we obtain

$$v''' + 3\cot\theta v'' + v'\left(4k - \csc^2\theta + 2\cot^2\theta\right) + 4v\left(\alpha^2\cot\theta\csc^2\theta + k\cot\theta\right) = 0. \tag{55}$$

Now, we make the change of variables $v \to f$ given by

$$f(\theta) = \sin^{4-n}\theta\, v(\theta)$$

in equation (55) and multiply the resulting equation through by $\sin^{n-4}\theta$. Setting $\alpha = (2-n)/2$, $\ell = \frac{1}{2}\left(\sqrt{4\lambda + (n-1)^2} - 1\right)$ (which is the positive solution of $4\ell(\ell+1) = 4\lambda + n^2 - 2n$) and, using (54) we see that f satisfies (52). This finishes the proof of Claim 4.3. $\square$

Lemma 4.4. *Let $\alpha = (2-n)/2$, $\ell = \frac{1}{2}\left(\sqrt{4\lambda + (n-1)^2} - 1\right)$, and μ be the first positive value of ℓ for which $P_\ell^\alpha(cos\theta_1)$ vanishes. Consider*

$$B \equiv h\sin^{n-1}\theta + \frac{g'\sin^{2n-2}\theta}{p+1} + \frac{(2n-2)g\sin^{2n-3}\theta\cos\theta}{p+1}, \tag{56}$$

where $h(\theta) = \frac{1}{2}g'\sin^{n-1}\theta$, $g(\theta) = f(\theta)\sin^{-2}\theta$ and $f(\theta) = \sin^{4-n}\theta P_\ell^\alpha(\cos\theta)P_\ell^{-\alpha}(\cos\theta)$. Then B is negative on $[0, \mu)$.

Proof. The associated Legendre functions satisfy the following raising and lowering relations (see, e.g., [1], equation 8.1.2, pp. 332), which we will use repeatedly in the proof of this lemma.

$$\dot{P}_\ell^\alpha(\cos\theta) = \frac{-P_\ell^{\alpha+1}}{\sin\theta} - \frac{\alpha\cos\theta P_\ell^\alpha}{\sin^2\theta} \tag{57}$$

and

$$\dot{P}_\ell^{\alpha+1}(\cos\theta) = \frac{1}{\sin^2\theta}\Big((\ell+\alpha+1)(\ell-\alpha)\sin\theta P_\ell^\alpha + (\alpha+1)\cos\theta P_\ell^{\alpha+1}\Big). \tag{58}$$

Notice that in the two previous equations, $\dot{P}_\ell^\alpha$ means the derivative of P_ℓ^α with respect to its argument, therefore,

$$\frac{d}{d\theta}P_\ell^\alpha(\cos\theta) = -\sin\theta\,\dot{P}_\ell^\alpha(\cos\theta).$$

After substituting for h, g and f we can write

$$B = -\sin\theta\frac{n-1}{n}\sin^n\theta(\dot{P}_\ell^\alpha P_\ell^{-\alpha} + P_\ell^\alpha\dot{P}_\ell^{-\alpha}). \tag{59}$$

Hence, it suffices to show that $\dot{P}_\ell^\alpha P_\ell^{-\alpha} + P_\ell^\alpha\dot{P}_\ell^{-\alpha} > 0$ on $[0,\mu)$. Because of Lemma 2.1, $P_\ell^\alpha P_\ell^{-\alpha}$ is positive, on this interval. Thus, we can write this inequality as

$$\frac{\dot{P}_\ell^\alpha}{P_\ell^\alpha} + \frac{\dot{P}_\ell^{-\alpha}}{P_\ell^{-\alpha}} > 0. \tag{60}$$

It follows from equation (57) that

$$\frac{\dot{P}_\ell^\alpha}{P_\ell^\alpha} + \frac{\dot{P}_\ell^{-\alpha}}{P_\ell^{-\alpha}} = -\frac{1}{\sin\theta}\frac{P_\ell^{\alpha+1}}{P_\ell^\alpha} - \frac{1}{\sin\theta}\frac{P_\ell^{-\alpha+1}}{P_\ell^{-\alpha}}. \tag{61}$$

Given, the identity (61) above, in order to prove (60) it is convenient to introduce the function,

$$y_\nu(\theta) = \frac{-1}{\sin\theta}\frac{P_\ell^{\nu+1}(\cos\theta)}{P_\ell^\nu(\cos\theta)} - \frac{\nu}{2\sin^2\frac{\theta}{2}} \tag{62}$$

In the sequel, we study the behavior of $y_\nu(\theta)$ on $[0,\mu)$. In particular, we will show that y_ν is positive on this interval if $-1<\nu<1$. This in turn will imply that

$$\frac{\dot{P}_\ell^\alpha}{P_\ell^\alpha} + \frac{\dot{P}_\ell^{-\alpha}}{P_\ell^{-\alpha}} = y_\alpha(\theta) + y_{-\alpha}(\theta) > 0.$$

The series expansion of the Legendre associated functions is given by the following expression:

$$P_\ell^\nu(\cos\theta) = \frac{1}{\Gamma(1-\nu)}\left(\cot\frac{\theta}{2}\right)^\nu {}_2F_1\left(-\ell,\ell+1,1-\nu,\sin^2\frac{\theta}{2}\right), \tag{63}$$

where

$${}_2F_1(\delta,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\delta)\Gamma(\beta)}\sum_{n=0}^\infty\frac{\Gamma(n+\delta)\Gamma(n+\beta)}{\Gamma(n+\gamma)n!}z^n. \tag{64}$$

We can write this last function as

$${}_2F_1(\delta,\beta,\gamma,z) = 1 + \frac{\delta\beta}{\gamma}z + \frac{\delta(\delta+1)\beta(\beta+1)}{2\gamma(\gamma+1)}z^2 + \mathcal{O}(z^3),$$

so that

$$P_\ell^\nu(\cos\theta) = \frac{1}{\Gamma(1-\nu)}\cot^\nu\frac{\theta}{2}\left(1 - \frac{\ell(\ell+1)}{1-\nu}\sin^2\frac{\theta}{2} + \frac{\ell(\ell^2-1)(\ell+2)}{2(1-\nu)(2-\nu)}\sin^4\frac{\theta}{2} + \mathcal{O}\left(\sin^6\frac{\theta}{2}\right)\right).$$

It follows that

$$\frac{P_\ell^{\nu+1}(\cos\theta)}{P_\ell^\nu(\cos\theta)} = \frac{\Gamma(1-\nu)}{\Gamma(-\nu)}\cot\frac{\theta}{2}\left(1 + E\sin^2\frac{\theta}{2} + \mathcal{O}\left(\sin^4\frac{\theta}{2}\right)\right), \tag{65}$$

where

$$E = \frac{\ell(\ell+1)}{\nu(1-\nu)}$$

(Here we used that $\Gamma(1-\nu) = -\nu\Gamma(-\nu)$. Thus, it follows from equations (62) and (65) that

$$y_\nu(\theta) = \frac{\nu}{2}\left(E + F\sin^2\frac{\theta}{2} + \mathcal{O}\left(\sin^4\frac{\theta}{2}\right)\right).$$

In particular,

$$\lim_{\theta\to 0} y_\nu(\theta) = \frac{\ell(\ell+1)}{2(1-\nu)} > 0,$$

since we are considering $\ell > 0$ and $-1 < \nu < 1$. We will now show by contradiction that there is no point on the interval $[0,\mu)$ where y_ν changes sign. To do so, we first derive a Riccati equation for y_ν. It follows from equation (62) that

$$\dot{y}_\nu = \frac{\cos\theta}{\sin^2\theta}\frac{P_\ell^{\nu+1}}{P_\ell^\nu} + \frac{\dot{P}_\ell^{\nu+1}}{P_\ell^\nu} - \frac{P_\ell^{\nu+1}\dot{P}_\ell^\nu}{(P_\ell^\nu)^2} + \frac{\nu(1+\cos\theta)^2}{\sin^3\theta}. \tag{66}$$

Using equations (57) and (58) in equation (66) we obtain

$$\dot{y}_\nu = \frac{1}{\sin\theta}\left(\frac{P_\ell^{\nu+1}}{P_\ell^\nu}\right)^2 + \frac{2(\nu+1)\cos\theta}{\sin^2\theta}\frac{P_\ell^{\nu+1}}{P_\ell^\nu} + \frac{(\ell+\nu+1)(\ell-\nu)}{\sin\theta} + \frac{\nu(1+\cos\theta)^2}{\sin^3\theta}. \tag{67}$$

Finally, using equation (62) to solve for $P_\ell^{\nu+1}/P_\ell^\nu$ we obtain the following Riccati equation for y_ν,

$$\dot{y}_\nu = \sin\theta y_\nu^2 + \frac{2y_\nu}{\sin\theta}(\nu - \cos\theta) + \frac{\ell(\ell+1)}{\sin\theta}. \tag{68}$$

Since $y_{\nu(0)} > 0$, and $y_\nu(\theta)$ is continuous in θ, If $y_\nu(\theta)$ were to cross $y_\nu = 0$, there would exist a point, θ^*, such that $y_\nu(\theta^*) = 0$ and $\dot{y}_\nu(\theta^*) < 0$. But from equation (68) we would then have

$$\dot{y}_\nu(\theta^*) = \frac{\ell(\ell+1)}{\sin\theta^*} > 0,$$

arriving at a contradiction. We conclude that y_ν is positive on $[0,\mu)$. □

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